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Bose–Einstein condensation of spin-1 field in an Einstein universe

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Abstract. In this paper we investigate the Bose–Einstein condensation of massive spin-1 particles in an Einstein universe. The system is considered under relativistic conditions taking into consideration the possibility of particle–antiparticle pair production. An exact expression for the charge density is obtained, then certain approximations are employed in order to obtain the solutions in closed form. A discussion of the approximations employed in this and other work is given. The effects of finite-size and spin-curvature coupling are emphasized.

1. Introduction

Since the early work of Altaie (1978), the study of Bose–Einstein condensation (BEC) in curved space has attracted the interest of many authors (Aragao de Carvalho and Goulart Rosa 1980, Singh and Pathria 1984, Parker and Zhang 1991, Smith and Toms 1996, Trucks 1998). This interest arises in the context of trying to understand the thermodynamics of the early universe and the role played by the finiteness of the space in determining the thermal behaviour of bosons in the respective systems.

Most studies employ the static Einstein universe as the underlying geometry for the system, since the thermodynamic equilibrium in this universe can be defined without ambiguity (Altaie and Dowker 1978). In a time developing spacetime no such luxury is enjoyed, except for the Robertson–Walker spacetime, which is conformally static (Kennedy 1978).

In an earlier work (Altaie 1978), we considered the cases of non-relativistic BEC of spin-0 and spin-1 bosons in an Einstein universe. We found that the finiteness of the system resulted in 'smoothing out' the singularities of the thermodynamic functions found in the infinite systems, the enhancement of the condensate fraction and the displacement of the specific-heat maximum toward higher temperatures.

Aragao de Carvalho and Goulart Rosa (1980) considered the case of a relativistic scalar field in an Einstein universe in an effort to show that the finite-size effects are negligible in comparison with the relativistic effects. However, a close look at their work shows that this comparison was made at an implicitly large value of the radius a of the Einstein universe, and since finite-size effects are proportional to $1/a$ the finite-size effects were found to be negligible. Moreover, the consideration of relativistic and ultra-relativistic BEC should take into account the possibility of particle–antiparticle production, since, at temperatures greater than the rest mass of the particles, quantum field theory requires the inclusion of such a process (Haber and Weldon 1981, 1982).

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Singh and Pathria (1984) considered the BEC of a relativistic conformally coupled scalar field in the Einstein universe and found qualitative and quantitative agreement with Altaie (1978). Parker and Zhang (1991) considered the ultra-relativistic BEC of the minimally coupled scalar field in an Einstein universe in the limit of high temperatures. They showed, among other things, that ultra-relativistic BEC can occur at very high temperature and densities in the Einstein universe, and by implication in the early stages of a dynamically changing universe. Parker and Zhang (1993) also showed that the BE condensate could act as a source for inflation leading to a de Sitter type universe.

Trucks (1998) repeated the calculations of Singh and Pathria (1984) but this time for the minimally coupled scalar field, obtaining similar results. In fact performing the calculations for the minimally coupled scalar field amounts to substituting $\overline{m} = (m^2 + 1/a^2)^{1/2}$ where m is the mass of the conformally coupled field. However, as the calculations were effectively considered in the large-radius region, the results come out to be identical with those of Singh and Pathria (1984).

The importance of the study of BEC in curved spaces stems from the interest in understanding the thermodynamics of the very early universe and that such a phenomenon may shed some light on the problem of mass generation in the very early universe. Indeed the investigations of Toms (1992, 1993) have shown that a kind of symmetry breaking is possible. However, in a latter paper Smith and Toms (1996) examined the BEC as a symmetry breaking in the specific model of the Einstein static universe. They claim that symmetry breaking never occurs in the sense that the chemical potential μ never reaches its critical value, but they also mention, toward the end of their paper, that if the volume of the universe is large then the value of μ is expected to be quite close to its critical value for some finite temperature; thus there will be a large number of particles in the ground state. Indeed this is the case considered in this paper, where we investigate the BEC at a relatively large-radius region and adopt the criterion that the onset of BEC will take place whenever a large number of particles is present at the ground state. Therefore, the limitations of Smith and Toms are avoided.

In this paper we will consider the onset of BEC of the relativistic spin-1 particle–antiparticle system in an Einstein universe, a state which is surly relevant for the early stages of the universe at a point when the electromagnetic interactions decouple from the weak interactions. We will carry out the calculations in a fashion similar to that of Singh and Pathria and compare the results with our earlier non-relativistic case. Throughout this work we adopt the absolute system of units in which $c = G = k = \hbar = 1$.

2. The charge density

We consider an ideal relativistic Bose gas of spin 1 confined to the background geometry of the spatial section S^3 of an Einstein universe with radius a. The metric of the Einstein static universe is given by

$$ds^2 = dt^2 - a^2[d\chi^2 + \sin^2\chi\,(d\theta^2 + \sin^2\theta\,d\phi^2)] \tag{1}$$

where $0 \leqslant \chi \leqslant \pi$, $0 \leqslant \theta \leqslant \pi$ and $0 \leqslant \phi \leqslant 2\pi$. The field equation for a non-interacting massive spin-1 field is given by

$$\nabla_\mu F^{\mu\nu} - m^2 A^\nu = 0 \tag{2}$$

where $F_{\mu\nu} = \nabla_\mu A_\nu - \nabla_\nu A_\mu$ is the field tensor, A_μ is the field 4-vector and ∇_μ is the covariant derivative. Since we are considering a relativistic system, it is necessary to take into consideration the possibility of pair production. For a particle–antiparticle system in thermal equilibrium at a finite temperature T the grand partition function is given by

$$Z = \mathrm{Tr}\exp[-\beta(\hat{H} - \mu\hat{Q})] \tag{3}$$

where $\hat{H}$ is the Hamiltonian operator of the system, $\hat{Q}$ is the total charge number operator and $\beta = 1/T$. The thermodynamic potential Ω of the system (excluding the vacuum contribution) is given by Haber and Weldon (1982)

$$\beta\Omega = \sum_k \ln[1 - \exp[-\beta(\epsilon_k - \mu)]] + \sum_k \ln[1 - \exp[-\beta(\epsilon_k + \mu)]] \tag{4}$$

where ϵ_k are the eigenenergies of the field. The net charge of the system is given by

$$Q = -\left[\frac{\partial\Omega}{\partial\mu}\right]_{T,V} = N_1 - N_2 \tag{5}$$

where

$$N_1 = \sum_n d_n[\mathrm{e}^{\beta(\epsilon_n-\mu)} - 1]^{-1} \qquad N_2 = \sum_n d_n[\mathrm{e}^{\beta(\epsilon_n+\mu)} - 1]^{-1} \tag{6}$$

are the average number of particles and antiparticles, respectively and d_n is the degeneracy of the nth level. The equation of motion of the spin-1 field in an Einstein universe was considered by Schrödinger (1938) and the solution yields the following energy spectrum:

$$\epsilon_n = \frac{1}{a}(n^2 + m^2a^2)^{1/2} \tag{7}$$

with degeneracy

$$d_n = 2(n^2 - 1) \qquad n = 2, 3, 4, \ldots . \tag{8}$$

The charge density q is then found to be

$$q = \frac{Q}{V} = \frac{4}{V}\sum_{l=1}^{\infty}\sum_{n=1}^{\infty}(n^2 - 1)\sinh(l\beta\mu)\exp[-l\beta'(m^2a^2 + n^2)^{1/2}] \tag{9}$$

where $V = 2\pi^2a^3$ is the volume of the spatial section of the Einstein universe, and $\beta' = 1/Ta$. In order to carry out the summation over n in (9) we apply the Poisson summation formula (see Titchmarsh 1948),

$$\sum_{n=1}^{\infty} f(n) + \tfrac{1}{2}f(0) = \int_0^{\infty} f(t)\,\mathrm{d}t + 2\sum_{j=1}^{\infty}\int_0^{\infty} f(t)\cos(2\pi jt)\,\mathrm{d}t. \tag{10}$$

Accordingly,

$$\sum_{n=1}^{\infty}(n^2 - 1)\exp[-l\beta'(m^2a^2 + n^2)^{1/2}] = \tfrac{1}{2}\exp(l\beta m) + I_0 + 2\sum_j I_j \tag{11}$$

where

$$I_0 = \int_0^{\infty}(t^2 - 1)\exp[-l\beta'(m^2a^2 + t^2)^{1/2}]\,\mathrm{d}t \tag{12}$$

and

$$I_j = \int_0^{\infty}(t^2 - 1)\exp[-l\beta'(m^2a^2 + t^2)^{1/2}]\cos(2\pi jt)\,\mathrm{d}t. \tag{13}$$

These integrals can be easily performed using (Gradshteyn and Ryzhik 1980)

$$\int_0^{\infty}\exp\left[-\alpha\sqrt{\gamma^2 + x^2}\right]\cos(\lambda x)\,\mathrm{d}x = \frac{\gamma\alpha}{\sqrt{\lambda^2 + \alpha^2}}K_1\left(\gamma\sqrt{\lambda^2 + \alpha^2}\right) \tag{14}$$

where K_ν are the modified Bessel functions of the second kind.

Evaluating I_0 and I_j and then substituting in (9) we find that the charge density can be written as

$$q = \frac{1}{2\pi^2 a^3}\left[\frac{1}{e^{\beta(m-\mu)}-1} - \frac{1}{e^{\beta(m+\mu)}-1}\right] - \frac{2m}{\pi^2 a^2}\sum_{l=1}^{\infty}\sum_{j=1}^{\infty} l\sinh(l\beta\mu)\left[\frac{K_1(l\beta m)}{l} + \frac{K_1(\beta m z)}{z}\right] + \frac{2m^2}{\pi^2\beta}\sum_{l=1}^{\infty}\sum_{j=-\infty}^{\infty} l\sinh(l\beta\mu)\left[\frac{K_2(\beta m z)}{z^2} - (\beta m j')^2\frac{K_3(\beta m z)}{z^3}\right] \tag{15}$$

where

$$z = \sqrt{l^2 + j'^2} \qquad \text{and} \qquad j' = (2\pi a/\beta) j. \tag{16}$$

The first term in (15) arises because the $n = 0$ term in the summation over n in (9) is non-zero. The second term arises because of the spin-curvature coupling, which is defined mathematically by the coefficient a_n of the Schwinger–De Witt expansion (see De Witt 1965). The third term is just twice that of the scalar case considered by Singh and Pathria (1984). Both the first and the second terms disappear in the limit $a \to \infty$. The bulk term is obtained in this limit assuming that q remains constant, which is the known thermodynamic limit. This reduces to the $j = 0$ contribution of the last term in (15), which gives

$$q_B(\beta, \mu) = \frac{2m^3}{\pi^2}\sum_{l=1}^{\infty}(l\beta m)^{-1}\sinh(l\beta\mu)K_2(l\beta m). \tag{17}$$

This is just twice the value obtained for the scalar case as would be expected. The summation over j in (15) can be performed using the Poisson summation formula (10) again, where we obtain

$$q = q_B(\beta,\mu) + \left[\frac{1}{e^{\beta(m-\mu)}-1} - \frac{1}{e^{\beta(m+\mu)}-1}\right] - \frac{2m}{\pi^2 a^2}\sum_{l=1}^{\infty}\sum_{p=1}^{\infty}\int_0^{\infty} l\sinh(l\beta\mu')\left[\frac{K_1(l\beta m)}{l} + \frac{K_1(\beta m z)}{z}\right]\mathrm{d}j + \frac{4m^2}{\pi^2\beta}\sum_{l=1}^{\infty}\sum_{p=1}^{\infty}\int_0^{\infty} l\sinh(l\beta\mu')\left[\frac{K_2(\beta m z)}{z^2} - (\beta m j)^2\frac{K_3(\beta m z)}{z^3}\right]\mathrm{d}j \tag{18}$$

where $\mu' = \mu + 2\pi i p$. The integrals in the above equation can be evaluated exactly using (see Singh and Pathria 1984)

$$\int_0^{\infty} j\sinh(jx)\frac{K_\nu[y(j^2+\xi^2)^{1/2}]}{(j^2+\xi^2)^{\nu/2}}\,\mathrm{d}j = \left(\frac{\pi\xi^3}{2}\right)^{1/2}\frac{x(y^2-x^2)^{\nu/2-3/4}}{(\xi y)^\nu}K_{\nu-3/2}[\xi(y^2-x^2)^{1/2}]. \tag{19}$$

Then using (see Gradshteyn and Ryzhik 1980)

$$\int_0^{\infty}\sinh(ax)K_1(bx)\,\mathrm{d}x = \frac{\pi}{2}\frac{a}{b\sqrt{b^2-a^2}} \tag{20}$$

and the relation

$$K_{1/2}(z) - zK_{3/2}(z) = -zK_{-1/2}(z) = -\left(\frac{\pi z}{2}\right)^{1/2}e^{-z} \tag{21}$$

we obtain

$$q = q_{\rm B}(\beta,\mu) + \frac{1}{2\pi^2 a^3}\left[\frac{1}{{\rm e}^{\beta(m-\mu)}-1} - \frac{1}{{\rm e}^{\beta(m+\mu)}-1}\right] - \frac{2}{\pi\beta}\sum_{l=1}^{\infty}\sum_{p=-\infty}^{\infty}\mu'\left[\frac{1}{2a^2\tau} + \left(\frac{1}{a^2\tau}+\tau\right){\rm e}^{-2\pi a l\tau}\right] \tag{22}$$

where $\tau = \sqrt{m^2-\mu'^2}$.

The summation over l can be easily performed, so the exact expression for the charge density becomes

$$q = q_{\rm B}(\beta,\mu) + \frac{1}{2\pi^2 a^3}\left[\frac{1}{{\rm e}^{\beta(m-\mu)}-1} - \frac{1}{{\rm e}^{\beta(m+\mu)}-1}\right] - \frac{2}{\pi\beta}\sum_{p=-\infty}^{\infty}\mu'\left[\frac{1}{2a^2\tau} + \left(\frac{1}{a^2\tau}+\tau\right)\frac{1}{{\rm e}^{2\pi a\tau}-1}\right]. \tag{23}$$

Note that this form of the charge density is exact and no approximation whatsoever has been made through the calculation.

3. Bose–Einstein condensation

We will adopt the microscopic criterion for marking the onset of the condensation (Altaie 1978), according to which the condensation region is defined such that a large number of particles is found occupying the ground state. This implies that the chemical potential μ of the system approaches the minimum single-particle energy ϵ_2, not the single-rest-mass energy considered by Singh and Pathria (1984). This can be observed directly from (6). However, the consideration of the criterion that $\mu \to m$ is justified only for the minimally coupled scalar field case where the minimum energy is m (see Parker and Zhang 1991). In our case the chemical potential must satisfy the condition that

$$-\left(m^2+\frac{4}{a^2}\right)^{1/2} < \mu < \left(m^2+\frac{4}{a^2}\right)^{1/2} \tag{24}$$

whereas in the conformally coupled spin-0 case the condition is

$$-\left(m^2+\frac{1}{a^2}\right)^{1/2} < \mu < \left(m^2+\frac{1}{a^2}\right)^{1/2}. \tag{25}$$

However, if m^2 is much larger than $1/a^2$ then it will be justified to take the limit $\mu \to m$, but this approximation may impose certain restrictions on the range of the region under consideration. We will follow Singh and Pathria (1984) and adopt such an approximation in this paper. In such a case the main contribution of the summation over p in (23) arises from the $p = 0$ term. Other terms are of the order of ${\rm e}^{-a\sqrt{m/\beta}}$, i.e. ${\rm O}({\rm e}^{-a/\lambda_T})$ where $\lambda_T = \sqrt{2\pi\beta/m}$ is the mean thermal wavelength of the particle. The bulk term will reduce to (Singh and Pandita 1983)

$$q_{\rm B}(\beta,\mu) = q_{\rm B}(\beta,m) - \frac{m}{\pi\beta}(m^2-\mu^2)^{1/2} + {\rm O}(m^2-\mu^2). \tag{26}$$

Therefore we can write

$$q_{\rm B}(\beta,\mu) \approx q_{\rm B}(\beta,m) - \frac{m}{\pi\beta}(m^2-\mu^2)^{1/2} - \frac{2\mu}{\pi\beta}\left[(m^2-\mu^2)^{1/2} - \frac{1}{a^2}(m^2-\mu^2)^{-1/2}\right]\frac{1}{\exp\left(2\pi a\sqrt{m^2-\mu^2}\right)-1} + \frac{1}{2\pi^2 a^3}\left[\frac{1}{{\rm e}^{\beta(m-\mu)}-1} - \frac{1}{{\rm e}^{\beta(m+\mu)}-1}\right]. \tag{27}$$

If we define the thermogeometric parameter y as

$$y = \pi a(m^2 - \mu^2)^{1/2} \tag{28}$$

then equation (27) becomes

$$q \approx q_{\mathrm{B}}(\beta, m) - \frac{m}{\pi\beta a}\left[\left(\frac{y}{\pi} + \frac{\pi}{y}\right)\coth y - \frac{\pi}{y^2}\right]. \tag{29}$$

From this equation the behaviour of the thermogeometric parameter y in the condensation region can be determined. It is clear that the second term in (29) defines the finite-size and spin-curvature effects.

4. The condensate fraction

The growth of the condensate fraction is studied here in comparison with the bulk case. This will show the finite-size and the spin-curvature effect in the range considered, i.e. $ma \gg 1$. The charge density in the ground state is obtained if we substitute $n = 2$ in (6). This gives

$$q_0 = \frac{6}{2\pi^2 a^3}[(\mathrm{e}^{\beta(\epsilon_2-\mu)} - 1)^{-1} - (\mathrm{e}^{\beta(\epsilon_2+\mu)} - 1)^{-1}]. \tag{30}$$

In the condensation region $\mu \to \epsilon_2$, so that the main contribution to the charge density in the ground state comes from the first term in the square bracket. This means that

$$q_0 \approx \frac{3}{\pi^2 a^3 \beta(\epsilon_2 - \mu)}. \tag{31}$$

From (28) as $\mu \to m$, we have

$$\mu \approx m\left(1 - \frac{y^2}{2\pi^2 a^2 m^2}\right). \tag{32}$$

On the same footing we can expand ϵ_2 as

$$\epsilon_2 \approx m\left(1 + \frac{2}{m^2 a^2}\right) \tag{33}$$

so that (31) becomes

$$q_0 \approx \frac{6m}{a\beta(y^2 + 4\pi^2)}. \tag{34}$$

This means that the macroscopic growth of the condensate will occur only when $y^2 \to -4\pi^2$ (i.e. $y \to 2\pi\mathrm{i}$). In order to see how this condensate compares with the bulk case we use the expansion

$$\coth y = \frac{1}{y} + 2y\sum_{k=1}^{\infty}\frac{1}{y^2 + \pi^2 k^2} \tag{35}$$

so that

$$\left(\frac{y}{\pi} + \frac{\pi}{y}\right)\coth y - \frac{\pi}{y^2} = \frac{5}{\pi} - \frac{6\pi}{y^2 + 4\pi^2} + \frac{2}{\pi}(y^2 + \pi^2)\sum_{k=3}^{\infty}\frac{1}{y^2 + \pi^2 k^2}. \tag{36}$$

Substituting this in (29) and using (34) we obtain

$$q_0 = q - q_{\mathrm{B}}(\beta, m) + \frac{2m}{\pi^2\beta a}\left[\frac{5}{2} + (y^2 + \pi^2)\sum_{k=3}^{\infty}\frac{1}{y^2 + \pi^2 k^2}\right]. \tag{37}$$

For the bulk system ($a \to \infty$) there exists a critical temperature, $T = T_c$, given by

$$q_B(\beta_c, m) = q. \tag{38}$$

This condition can be written as

$$q = q_B(\beta_c, m) = \frac{m^3}{\pi^2} \sum_{l=1}^{\infty} (l\beta_c m)^{-1} \sinh(l\beta_c m) K_2(l\beta_c m) = \frac{m^3}{2\pi^2} W(\beta_c, m) \tag{39}$$

where

$$W(\beta, \mu) = 2 \sum_{l=1}^{\infty} (l\beta\mu)^{-1} \sinh(l\beta\mu) K_2(l\beta\mu). \tag{40}$$

Thus we can write the bulk condensate density as

$$(q_0)_B = \begin{cases} 0 & \text{for} \quad T > T_c \\ q\left(1 - \dfrac{W(\beta, m)}{W(\beta_c, m)}\right) & \text{for} \quad T < T_c. \end{cases} \tag{41}$$

For the case of the finite system under consideration the condensate density is given by

$$q_0 = q\left[1 - \frac{W(\beta, m)}{W(\beta_c, m)}\right] + \frac{2m}{\pi^2 \beta a}\left[\frac{5}{2} + (y^2 + \pi^2) \sum_{k=3}^{\infty} \frac{1}{y^2 + \pi^2 k^2}\right] \tag{42}$$

where the y dependence on T in the condensation region can now be determined explicitly by

$$\left(\frac{y}{\pi} + \frac{\pi}{y}\right) \coth y - \frac{\pi}{y^2} = -\frac{\pi\beta a q}{m}\left[1 - \frac{W(\beta, m)}{W(\beta_c, m)}\right]. \tag{43}$$

This equation can be solved numerically for the values of y at different temperatures. However, we notice that at $T = T_c$ ($W(\beta, m) = W(\beta_c, m)$) the value of y can be obtained by solving the equation

$$\left(\frac{y_c}{\pi} + \frac{\pi}{y_c}\right) \coth y_c = \frac{\pi}{y_c^2} \tag{44}$$

which has a solution at $y_c = 4.859\mathrm{i}$.

5. Non- and ultra-relativistic limits

The non-relativistic limit is obtained by setting $\beta m \gg 1$. In this case we can use the asymptotic expansion of the Bessel functions of the second kind for large argument, where we have (see Abramowitz and Stegun 1968)

$$K_2(j\beta m) \approx \left(\frac{\pi}{2j\beta m}\right)^{1/2} \mathrm{e}^{-j\beta m}\left[1 + \frac{15}{8}\frac{1}{j\beta m} + \frac{105}{128}\frac{1}{(j\beta m)^2} + \cdots\right] \tag{45}$$

so that

$$[W(\beta, m)]_{\mathrm{NR}} = \left(\frac{\pi}{2m^3}\right)^{1/2} \zeta\left(\frac{3}{2}\right) \beta^{-3/2} \tag{46}$$

and

$$\left[\frac{W(\beta, m)}{W(\beta_c, m)}\right]_{\mathrm{NR}} = \left(\frac{\beta_c}{\beta}\right)^{3/2} = \left(\frac{T}{T_c}\right)^{3/2}. \tag{47}$$

Therefore from (41) we have for the bulk system

$$(q_0)_B = q\left[1 - \left(\frac{T}{T_c}\right)^{3/2}\right] \qquad T < T_c. \tag{48}$$

In the non-relativistic limit (43) reduces to

$$2\left(\frac{y}{\pi}+\frac{\pi}{y}\right)\coth y-\frac{2\pi}{y^2}=x^{1/2}[1-x^{-3/2}]\frac{a}{\bar{l}}\left[2\zeta\left(\frac{3}{2}\right)\right]^{2/3} \tag{49}$$

where $x = T/T_c$ and $\bar{l}$ is the mean inter-particle distance. Note that equation (49) is the corrected version of equation (62) of our earlier paper (Altaie 1978).

In the ultra-relativistic limit $\beta m \ll 1$; therefore, we use the expansion

$$K_2(j\beta m)\sim\tfrac{1}{2}\Gamma(2)(\tfrac{1}{2}j\beta m)^{-2} \tag{50}$$

so that

$$[W(\beta,m)]_{UR}\sim\frac{2\pi^2}{3m^2}\beta^{-2}. \tag{51}$$

Accordingly the ultra-relativistic behaviour of the bulk charge density given in (41) will be

$$(q_0)_B=q\left[1-\frac{T^2}{T_c^2}\right]. \tag{52}$$

However, substituting (51) in (39), the critical temperature for the bulk spin-1 particles is

$$T_c=\sqrt{\frac{3q}{2m}}. \tag{53}$$

This is the result analogous to that of the minimally coupled scalar field. This shows that the treatment of the problem using the large-radius approximation as adopted by Singh and Pathria (1984) and in this paper is equivalent to the high-temperature expansion of Parker and Zhang (1991). This can be understood in the light of the fact that in the ultra-relativistic regime $T \propto a^{-1}$, so aT = constant. In the ultra-relativistic regime the thermogeometric parameter y behaves according to

$$\left(\frac{y}{\pi}+\frac{\pi}{y}\right)\coth y=\frac{\pi aq}{mT_c}\left(x-\frac{1}{x}\right)=\pi a\sqrt{\frac{2q}{3m}}\left(x-\frac{1}{x}\right). \tag{54}$$

This equation can be solved for given values of a, q and m. For illustrative purposes, figure 1 shows the behaviour of the thermogeometric parameter y (drawn as $v = \sqrt{y^2+4\pi^2}$ versus the scaled temperature $x = T/T_c$). It is clear that as a increases the quantity $\sqrt{y^2+4\pi^2}$ tends to a step function.

6. The critical radius

Although it is known that the universe as a whole is neutral or almost neutral, the observational limit on the net average charge density does not exclude the possibility that the charge density in the early universe was sufficient to produce relativistic BEC. Following the assumption that the charge is conserved we can write

$$q_i=\left(\frac{a_p}{a_i}\right)^3 q_p \tag{55}$$

where q_i and q_p are the initial and the present charge densities, respectively. If we consider $Ta = C$ and assume that C remains constant throughout the development of the universe then from (53) we can allocate a critical radius for the universe a_c below which the gas will be always in the condensate state. Parker and Zhang (1991) have already noted this. However for the spin-1 case this critical radius will be given by

$$a_c=\frac{3q_p a_p^3}{2mC^2}. \tag{56}$$

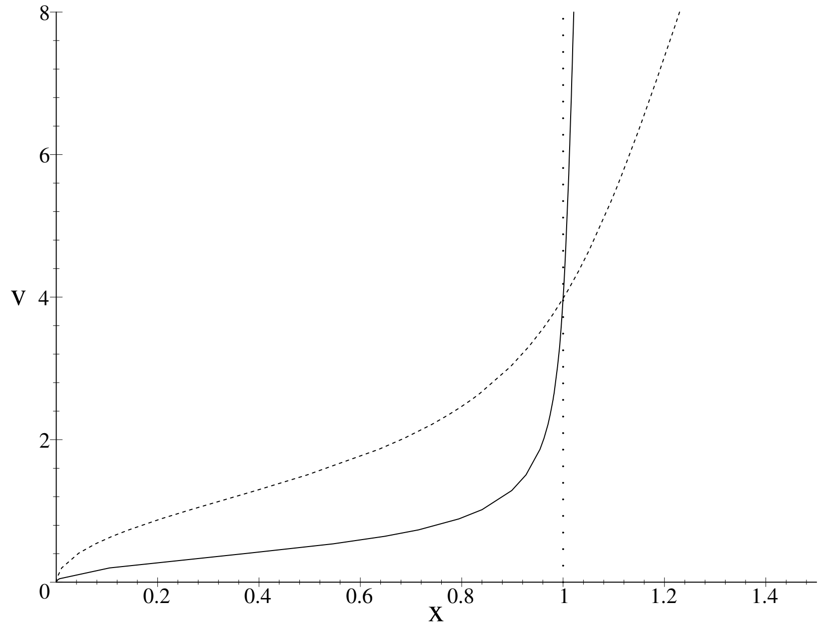

Figure 1. The quantity $v = \sqrt{y^2 + 4\pi^2}$, where y is the thermogeometric parameter, is plotted as a function of the scaled temperature $x = T/T_c$ for two different values of the radius a. The solid curve is for the larger value of a. The vertical dotted line at $x = 1$ is marked for reference.

The estimated upper bound on the net average charge density of the universe at present is $q < 10^{-24}$ cm^{-3} (see Dolgov and Zeldovich 1981). If this upper bound on the present charge density is adopted, then an upper bound on the radius of the universe at which the condensate starts can be deduced. Using a value of 10^{28} for C (Trucks 1998) and $a_p = 10^{28}$ cm, the critical radius for the onset of the condensation of the heavy gauge bosons $W^{\pm}$ can be calculated. This gives $a_c < 10^{-11}$ cm, which implies that when $a < a_c$ the charged boson system is in a condensate phase; above a_c the condensation temperature T_c becomes lower than the radiation temperature T so a phase transition occurs to the normal phase. However, having a finite-sized system the phase transition taking place is not critical since it is known that the thermodynamic functions are continuous around T_c. Above a_c the critical temperature is $T_c = 0$: at such a point the condensation is totally absent and the charged boson system becomes a relativistic matter system characterized by $T \propto a^{-1}$. Thus a_c marks an upper limit for the radius of the universe at which the condensation of massive spin-1 particles will be present.

7. Discussion

In the previous sections we have investigated the behaviour of an ideal relativistic massive spin-1 gas confined to the background geometry of an Einstein universe. We have found that there is a finite condensation temperature for the system of massive bosons and that the thermodynamics functions are regular throughout the range of temperature below and above the condensation temperature. The onset of condensation is marked by the presence of a large number of particles at the ground state of the system. It should be emphasized that finite-size effects smooth out the singularities of the corresponding infinite system. This in turn makes the phase transition non-critical. The large-radius approximation was adopted to obtain a closed-form result. However, since the finite-size effects are inversely proportional to the radius of the universe these effects are minimal in this case.

We have used the Poisson summation formula to evaluate the summations involved in the expression for the charge number density. This formula has already been extensively used in other similar works, in both the non-relativistic and relativistic cases. However, Smith and Toms (1996) pointed out that when using the Poisson summation formula one should be careful to ensure that the domain of convergence of the final result is the same as that of the original zeta function approach. This argument was brought up when they compared the domain of convergence of the effective action of the massive scalar field with the final result for the charged number density of the field. The difference between the two domains was shown to be an additive factor of $1/a^2$. In our opinion, the use of the Poisson summation formula is legitimate whenever the desired Fourier transforms of the functions exist. This is indeed the case in this paper and in the work of Singh and Pathria (1984). Moreover, the reservations of Smith and Toms (1996) are overtaken in the case of adopting the large-radius approximation, since in such a case the domain of convergence of results obtained via the Poisson summation formula is very close to that obtained using the zeta function approach.

The extension of the problem investigated in this paper to the Robertson–Walker universe is quite possible and we expect to obtain, qualitatively at least, similar results. This expectation is supported by the fact that the Robertson–Walker universe is conformally static. The question of thermal equilibrium in the Robertson–Walker universe was addressed by Kennedy (1978), who showed that the thermal Green functions for the static Einstein universe and the time-dependent Robertson–Walker universe are conformally related, hence deducing a (1–1) correspondence between the vacuum and the many particle states of the static Einstein universe and the Robertson–Walker universe.

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